

ON THE RADIAL SOLUTIONS OF A STATIONARY SCHRÖDINGER SYSTEM WITH A NONLINEAR RANDOM OPERATOR

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ABSTRACT. Our main goal is to establish sufficient conditions for the existence of positive entire radially symmetric solutions for a system of the type

$$\begin{cases} \operatorname{div}(\phi_1(|\nabla u|)\nabla u) + \sigma_1(|x|)\phi_1(|\nabla u|)|\nabla u| = p_1(|x|)f_1(u, v), & x \in \mathbb{R}^N, \\ \operatorname{div}(\phi_2(|\nabla v|)\nabla v) + \sigma_2(|x|)\phi_2(|\nabla v|)|\nabla v| = p_2(|x|)f_2(u, v), & x \in \mathbb{R}^N, \end{cases}$$

where $\phi_1, \phi_2, a_1, a_2, p_1, p_2, f_1$ and f_2 are continuous functions satisfying certain properties. Our results are obtained by an application of the Arzela–Ascoli theorem.

1. INTRODUCTION

The existence of positive entire solutions for the coupled nonlinear systems

$$\begin{cases} \Delta_{\phi_1} u + \sigma_1(|x|)\phi_1(|\nabla u|)|\nabla u| = p_1(|x|)f_1(u, v) \text{ for } x \in \mathbb{R}^N, \\ \Delta_{\phi_2} v + \sigma_2(|x|)\phi_2(|\nabla v|)|\nabla v| = p_2(|x|)f_2(u, v) \text{ for } x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where \mathbb{R}^N ($N \geq 3$) denote the Euclidean N -space, $|\cdot|$ will denote any N -dimensional norm, $\Delta_{\phi_i} w$ ($i = 1, 2$) stands for the ϕ_i -Laplacian operator defined as $\Delta_{\phi_i} w := \operatorname{div}(\phi_i(|\nabla w|)\nabla w)$ and the functions ϕ_i satisfy, throughout this paper:

- O1) $\phi_i \in C^1((0, \infty), (0, \infty))$,
- O2) $t\phi_i(t)$ is strictly increasing in $(0, \infty)$,
- O3) there exist $l_i, m_i > 1$ such that

$$\text{if } \Phi_i(t) := \int_0^t s\phi_i(s)ds \text{ then } l_i \leq \frac{\Phi_i'(t) \cdot t}{\Phi_i(t)} \leq m_i \text{ for any } t > 0,$$

- O4) there exist $a_0^i, a_1^i > 0$ such that

$$a_0^i \leq \frac{\Phi_i''(t) \cdot t}{\Phi_i'(t)} \leq a_1^i \text{ for any } t > 0,$$

have been intensively studied in the last few decades in view of the understanding of some basic phenomena arising in physics (for more details, see [2, 3], Kawano-Kusano [9], Franchi-Lanconelli-Serrin [7], Fukagai-Narukawa [8], Grosse-Martin [10] and Smooke [20]). Below there are some examples of functions φ_1 and ϕ_2

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that fulfil (O1)-(O4) which, cf. [8], arise in mathematical models in nonlinear physical science:

E1: Nonlinear Elasticity:

$$\Phi_i(t) = (1 + t^2)^p - 1, \quad \phi_i(t) = 2p(1 + t^2)^{p-1},$$

where $t > 0$ and $p > \frac{1}{2}$;

E2: Plasticity:

$$\Phi_i(t) = t^p (\ln(1 + t))^q, \quad \phi_i(t) = \frac{\ln^{q-1}(t + 1)}{t + 1} [(pt^{p-1} + pt^{p-2}) \ln(t + 1) + qt^{p-1}],$$

where $t > 0$, $p > 1$ and $q > 0$;

E3: Generalized Newtonian fluids:

$$\Phi_i(t) = \int_0^t s^{1-p} (\sinh^{-1} s)^q ds, \quad \phi_i(t) = t^{-p} \operatorname{arcsinh}^q t,$$

where $t > 0$, $0 \leq p \leq 1$ and $q > 0$;

E4: Plasma Physics:

$$\Phi_i(t) = \frac{t^p}{p} + \frac{t^q}{q}, \quad \phi_i(t) = t^{p-2} + t^{q-2},$$

where $t > 0$ and $1 < p < q$.

E5: Non-Newtonian Fluid:

$$\Phi_i(t) = \frac{t^p}{p}, \quad \phi_i(t) = t^{p-2},$$

where $t > 0$ and $p > 1$.

Remark 1.1. The systems of the form (1.1) are known today as coupled nonlinear systems of Schrödinger type. Since, in particular, one of the most important classes of (1.1) is the time-independent Schrödinger equation in quantum mechanics

$$\Delta u = \frac{8\pi^2 m}{h^2} (V(r) - E) u, \tag{1.2}$$

where m is the particle's "reduced mass", $V(r)$ is its potential energy, r is a position vector, h is the Planck constant, E is the energy of a photon and the unknown function u is the wave function (for more details, see [10], [15], [19], [20] and [23]).

In the literature, an *entire large solution* means a couple $(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$ of positive functions satisfying (1.1) and such that both $u(x)$ and $v(x)$ tend to infinity as $|x| \rightarrow \infty$; an *entire bounded solution* if the condition $u(x) < \infty$ and $v(x) < \infty$ as $|x| \rightarrow \infty$; a *semifinite entire large solution* when $(u(x) < \infty$ and $v(x)$ tend to infinity) or $(u(x)$ tend to infinity and $v(x) < \infty)$ as $|x| \rightarrow \infty$.

In the next, we shall reserve r for the polar distance, $r := \sqrt{x_1^2 + \dots + x_n^2}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^N$. Note that if ϕ_i is considered as a function in \mathbb{R}^N depending only on r , then

$$\Delta_{\phi_i} w = (\phi_i(|w'|) |w'|)' + \frac{N-1}{r} \phi_i(|w'|) |w'|.$$

We would like to quote some references where the existence of entire bounded radial solutions, or the existence of entire large radial solutions for the systems of the form (1.1) were analyzed. Lair, [13] has considered the entire large radial solutions for the elliptic system

$$\begin{cases} \Delta u = p_1(r) v^\alpha, \\ \Delta v = p_2(r) u^\beta, \end{cases} \quad r = |x|, \quad x \in \mathbb{R}^N \quad (N \geq 3), \quad (1.3)$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$, p_1 and p_2 are nonnegative continuous functions on \mathbb{R}^N . He proved that a necessary and sufficient condition for this system to have a positive entire large radial solution, is

$$\int_0^\infty t p_1(t) \left(t^{2-N} \int_0^t s^{N-3} Q(s) ds \right)^\alpha dt = \infty, \quad (1.4)$$

$$\int_0^\infty t p_2(t) \left(t^{2-N} \int_0^t s^{N-3} P(s) ds \right)^\beta dt = \infty, \quad (1.5)$$

where $P(r) = \int_0^r \tau p_1(\tau) d\tau$ and $Q(r) = \int_0^r \tau p_2(\tau) d\tau$.

It is well known, see Yang [22], that if $p : [0, \infty) \rightarrow [0, \infty)$ is a spherically symmetric continuous function and the nonlinearity $f : [0, \infty) \rightarrow [0, \infty)$ is a continuous, increasing function with $f(0) \geq 0$ and $f(s) > 0$ for all $s > 0$ which satisfies

$$\int_1^\infty \frac{1}{f(t)} dt = \infty, \quad (1.6)$$

then, the single equation

$$\Delta u = p(r) f(u) \quad \text{for } x \in \mathbb{R}^N \quad (N \geq 3), \quad \lim_{r \rightarrow \infty} u(r) = \infty, \quad (1.7)$$

has a nonnegative radial solution if and only if p satisfies

$$\lim_{r \rightarrow \infty} \mathcal{P}_p(r) = \infty, \quad \mathcal{P}_p(r) := \int_0^r s^{1-N} \int_0^s z^{N-1} p(z) dz ds.$$

A direct computation gives

$$\lim_{r \rightarrow \infty} \mathcal{P}_p(r) = \frac{1}{N-2} \int_0^\infty t p(t) dt.$$

However, there is no existence results for the system (1.1) where f_1 and f_2 satisfy a condition of the form (1.6). This observation can be found in the paper of [13]. Fang-Yi [6], in the particular case $\varphi_1(t) = \varphi_2(t) = t^{p-1}$ ($p > 1$), supplied a sufficient condition

$$\int_a^\infty \frac{1}{f_1^{1/(p-1)}(t, t) + f_2^{1/(p-1)}(t, t)} dt = \infty, \quad t \geq a > 0, \quad (1.8)$$

for the existence of positive radial large solutions to (1.1). The condition (1.8) have been used by many authors and in many contexts, see Li-Zhang-Zhang [14], Liu-Zhang [16], Qin-Yang [21], Dkhil-Zeddini [5], and the references therein.

Now we return to (1.8): if the function $(f_1^{1/(p-1)} + f_2^{1/(p-1)})$ satisfies condition (1.8), so do separately each of the functions $f_1^{1/(p-1)}$ and $f_2^{1/(p-1)}$ but the converse is not true as one can see from the paper of Bernfeld [1, Example 3.8., pp. 283].

One of our main purposes of this paper is to establish sufficient conditions for the existence of entire large radial solutions of the system (1.1) under the new conditions of the form

$$\int_a^\infty \frac{1}{f_1^{1/(p-1)}(t, t)} dt = \infty, \quad t \geq a > 0, \quad (1.9)$$

and

$$\int_a^\infty \frac{1}{f_2^{1/(p-1)}(t, t)} dt = \infty, \quad t \geq a > 0, \quad (1.10)$$

The existence of entire bounded/semifinite entire large positive solutions is also studied in this paper. Finally, we should like to mention that the method presented here also yields much more precise information on the behavior of solutions.

2. NOTATIONS AND PRELIMINARIES

We work under the following assumptions:

- (P1) $\sigma_1, \sigma_2, p_1, p_2 : [0, \infty) \rightarrow [0, \infty)$ are continuous functions;
- (C1) $f_1, f_2 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ are continuous, increasing, $f_1(0, 0) \geq 0$, $f_2(0, 0) \geq 0$ and $f_1(s_1, s_2) > 0$, $f_2(s_1, s_2) > 0$ whenever $s_1, s_2 > 0$;
- (C2) there exist the continuous and increasing functions $h_1, h_2 : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ and $\bar{f}_1, \bar{f}_2 : [0, \infty) \rightarrow [0, \infty)$ such that

$$f_1(t_1, t_1 \cdot s_1) \leq h_1(t_1, t_1) \cdot \bar{f}_1(s_1), \quad \forall s_1 \geq 1 \text{ and } \forall t_1 \geq M_1 a_1, \quad (2.1)$$

$$f_2(t_2, t_2 \cdot s_2) \leq h_2(t_2, t_2) \cdot \bar{f}_2(s_2), \quad \forall s_2 \geq 1 \text{ and } \forall t_2 \geq M_2 a_2, \quad (2.2)$$

where $a_1, a_2 \in (0, \infty)$, $M_1 \geq \max\left\{1, \frac{1}{a_1}\right\}$ and $M_2 \geq \max\left\{1, \frac{1}{a_2}\right\}$.

In order to state our existence theorems, we introduce the following notations

$$\begin{aligned} Z(r) &= \int_{a_1+a_2}^r \frac{1}{\bar{\theta}_1(f_1(t, t)) + \bar{\theta}_2(f_2(t, t))} dt, \quad \mathcal{H}_i(r) = \int_{a_i}^r \frac{1}{\bar{\theta}_i(h_i(t, M_i t))} dt, \\ \xi_i(t) &= t^{N-1} e^{\int_0^t \sigma_i(s) ds}, \quad P_i(r) = \int_0^r \Psi_i^{-1} \left(\frac{1}{\xi_i(z)} \int_0^z \xi_i(t) p_i(t) dt \right) dz, \\ \bar{P}_i(r) &= \int_0^r \Psi_i^{-1} \left(\frac{1}{\xi_i(t)} \int_0^t \xi_i(s) p_i(s) \bar{f}_i(1 + Z^{-1}(P_1(s) + P_2(s))) ds \right) dt, \\ \underline{P}_1(r) &= \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(a_1, a_2 + \underline{\theta}_2(f_2(a_1, a_2))) P_2(s) ds \right) dt, \\ \underline{P}_2(r) &= \int_0^r \Psi_2^{-1} \left(\frac{1}{\xi_2(t)} \int_0^t \xi_2(s) p_2(s) f_2(a_1 + \underline{\theta}_1(f_1(a_1, a_2) P_1(s), a_2) ds \right) dt, \\ \mathcal{H}_i(\infty) &= \lim_{s \rightarrow \infty} \mathcal{H}_i(s), \quad P_i(\infty) = \lim_{r \rightarrow \infty} P_i(r), \quad i = 1, 2, \\ \bar{P}_i(\infty) &= \lim_{r \rightarrow \infty} \bar{P}_i(r), \quad \underline{P}_i(\infty) = \lim_{r \rightarrow \infty} \underline{P}_i(r). \end{aligned}$$

Some remarks are now in order on the preliminaries stated in this and the previous section.

Remark 2.1. A simple example of f_1 and f_2 satisfying (C2) is given by $f_1(u, v) = u^{\beta_1} v^{\alpha_1}$ and $f_2(u, v) = u^{\beta_2} v^{\alpha_2}$ with $\alpha_1, \beta_1, \alpha_2, \beta_2 \in [0, \infty)$ with $\alpha_1^2 + \beta_1^2 \neq 0$ and $\alpha_2^2 + \beta_2^2 \neq 0$.

For the proof of the next remark that will be stated below, we refer the reader to [8, Lemma 2.1].

Remark 2.2. Suppose ϕ_i ($i = 1, 2$) satisfy (O1), (O2), (O3) and (O4). Then,

$$\underline{\theta}_i(s_1)\Psi_i^{-1}(s_2) \leq \Psi_i^{-1}(s_1 s_2) \leq \bar{\theta}_i(s_1)\Psi_i^{-1}(s_2) \text{ for all } s_1, s_2 > 0, \quad (2.3)$$

where $\underline{\theta}_i(t) = \min \{t^{1/m_i}, t^{1/l_i}\}$, $\bar{\theta}_i(t) = \max \{t^{1/m_i}, t^{1/l_i}\}$.

The reader is referred to Krasnosel'skii and Rutickii [12] (see also Rao and Ren [18]) for a through treatment of the assumptions (C2) and (2.3).

Remark 2.3. If $P_i(\infty) = \infty$ then $\underline{P}_i(\infty) = \infty$ and $\bar{P}_i(\infty) = \infty$. On the other hand, if $\underline{P}_i(\infty) = \infty$ or $\bar{P}_i(\infty) = \infty$ then we can have one of the following

1. $P_1(\infty) < \infty$ and $P_2(\infty) = \infty$,
2. $P_1(\infty) = \infty$ and $P_2(\infty) < \infty$,
3. $P_1(\infty) = \infty$ and $P_2(\infty) = \infty$,

(see [13] for an example in this direction).

3. STATEMENTS AND PROOFS OF THE THEOREMS

Our main objective in this work is to prove the following result:

Theorem 3.1. *The system (1.1) has one positive radial solution $(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$ given that $\mathcal{H}_1(\infty) = \mathcal{H}_2(\infty) = \infty$ and (P1), (C1), (C2) hold true. Moreover, if $\underline{P}_1(\infty) = \infty$ and $\underline{P}_2(\infty) = \infty$ then*

$$\lim_{r \rightarrow \infty} u(r) = \infty \text{ and } \lim_{r \rightarrow \infty} v(r) = \infty.$$

Proof of Theorem 3.1: We start by showing that (1.1) has positive radial solutions. On this purpose we can see that radial solutions of the system

$$\begin{cases} (\phi_1(u')u')' + \frac{N-1}{r}\phi_1(u')u' + \sigma_1(r)\phi_1(u')u' = p_1(r)f_1(u(r), v(r)), & r \geq 0, \\ (\phi_2(v')v')' + \frac{N-1}{r}\phi_2(v')v' + \sigma_2(r)\phi_2(v')v' = p_2(r)f_2(u(r), v(r)), & r \geq 0, \\ u', v' \geq 0 \text{ on } [0, \infty) \\ u(0) = a_1, v(0) = a_2 \end{cases} \quad (3.1)$$

solve (1.1). By the symmetry of (u, v) and using the standard integrating procedure, we rewrite the system (3.1) as

$$\begin{cases} u(r) = a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(u(s), v(s)) ds \right) dt, & r \geq 0, \\ v(r) = a_2 + \int_0^r \Psi_2^{-1} \left(\frac{1}{\xi_2(t)} \int_0^t \xi_2(s) p_2(s) f_2(u(s), v(s)) ds \right) dt, & r \geq 0. \end{cases} \quad (3.2)$$

The solution (u, v) can be constructed by the following approximate scheme: define $u_0 = a_1, v_0 = a_2$ and let $\{(u_n, v_n)\}_{n \geq 1}$ on $[0, \infty) \times [0, \infty)$ given by

$$\begin{cases} u_n(r) = a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(u_{n-1}(s), v_{n-1}(s)) ds \right) dt, & r \geq 0, \\ v_n(r) = a_2 + \int_0^r \Psi_2^{-1} \left(\frac{1}{\xi_2(t)} \int_0^t \xi_2(s) p_2(s) f_2(u_{n-1}(s), v_{n-1}(s)) ds \right) dt, & r \geq 0. \end{cases} \quad (3.3)$$

We show that $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ are nondecreasing on $[0, \infty)$. To see this, express

$$\begin{aligned} u_1(r) &= a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(u_0(s), v_0(s)) ds \right) dt \\ &= a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(a_1, a_2) ds \right) dt \\ &\leq a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(u_1(s), v_1(s)) ds \right) dt = u_2(r). \end{aligned}$$

This proves that $u_1(r) \leq u_2(r)$. Similarly, $v_1(r) \leq v_2(r)$. By an induction argument we get

$$u_n(r) \leq u_{n+1}(r) \text{ for any } n \in \mathbb{N} \text{ and } r \in [0, \infty),$$

and

$$v_n(r) \leq v_{n+1}(r) \text{ for any } n \in \mathbb{N} \text{ and } r \in [0, \infty).$$

Let us now prove that the non-decreasing sequences $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ are bounded from above on bounded sets. By the monotonicity of $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ one get

$$\begin{aligned} [\xi_1(r) \phi_1(u'_n(r)) u'_n(r)]' &= \xi_1(r) p_1(r) f_1(u_{n-1}(r), v_{n-1}(r)) \\ &\leq \xi_1(r) p_1(r) f_1(u_n(r), v_n(r)), \end{aligned} \quad (3.4)$$

$$[\xi_2(r) \phi_2(v'_n(r)) v'_n(r)]' \leq \xi_2(r) p_2(r) f_2(u_n(r), v_n(r)). \quad (3.5)$$

Integrating the above inequalities and using (2.3), yield that

$$\begin{aligned} u'_n(r) &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1(u_n(s), v_n(s)) ds \right) \\ &\leq \Psi_1^{-1} \left(\frac{f_1(u_n(r), v_n(r))}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) ds \right) \\ &\leq \bar{\theta}_1(f_1(u_n(r), v_n(r))) \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) ds \right) \\ &= \bar{\theta}_1(f_1(u_n(r) + v_n(r), u_n(r) + v_n(r))) P'_1(r), \end{aligned}$$

and

$$v'_n(r) \leq \bar{\theta}_2(f_2(u_n(r) + v_n(r), u_n(r) + v_n(r))) P'_2(r).$$

It follows from these last inequalities that

$$\frac{(u_n(r) + v_n(r))'}{(\bar{\theta}_1(f_1) + \bar{\theta}_2(f_2))((u_n(r) + v_n(r), u_n(r) + v_n(r)))} \leq P'_1(r) + P'_2(r), \quad (3.6)$$

from which inequality we obtain

$$\int_{a_1+a_2}^{u_n(r)+v_n(r)} \frac{1}{\bar{\theta}_1(f_1(t,t)) + \bar{\theta}_2(f_2(t,t))} dt \leq P_1(r) + P_2(r).$$

Now we have

$$Z(u_n(r) + v_n(r)) \leq P_1(r) + P_2(r), \quad (3.7)$$

which will play a basic role in the proof of our main results. The inequalities (3.7) can be rewritten as

$$u_n(r) + v_n(r) \leq Z^{-1}(P_1(r) + P_2(r)). \quad (3.8)$$

This can be easily seen from the fact that Z is a bijection with the inverse function Z strictly increasing on $[0, Z(\infty))$. Let $M_1 \geq \max\left\{1, \frac{1}{a_1}\right\}$ and $M_2 \geq \max\left\{1, \frac{1}{a_2}\right\}$. The next step is to integrate (3.4) from 0 to r and bearing in mind (2.1), we find

$$\begin{aligned} (u_n(r))' &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1(u_n(s), v_n(s)) ds \right) \\ &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1(u_n(s), 2u_n(s) + v_n(s)) ds \right) \\ &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1(u_n(s), u_n(s) + Z^{-1}(P_1(s) + P_2(s))) ds \right) \\ &= \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1 \left(u_n(s), u_n(s) \left(1 + \frac{1}{u_n(s)} Z^{-1}(P_1(s) + P_2(s)) \right) \right) ds \right) \\ &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1 \left(u_n(s), u_n(s) \left(1 + \frac{1}{a_1} Z^{-1}(P_1(s) + P_2(s)) \right) \right) ds \right) \quad (3.9) \\ &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1(u_n(s), M_1(1 + Z^{-1}(P_1(s) + P_2(s)))) ds \right) \\ &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} h_1(u_n(r), M_1 u_n(r)) \int_0^r \xi_1(s) p_1(s) \bar{f}_1(1 + Z^{-1}(P_1(s) + P_2(s))) ds \right) \\ &\leq \bar{\theta}_1(h_1(u_n(r), M_1 u_n(r))) \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) \bar{f}_1(1 + Z^{-1}(P_1(s) + P_2(s))) ds \right) \\ &= \bar{\theta}_1(h_1(u_n(r), M_1 u_n(r))) \bar{P}'_1(r). \end{aligned}$$

Dividing the inequality (3.9) by $\bar{\theta}_1(h_1(u_n(r), M_1 u_n(r)))$ we see that

$$\frac{(u_n(r))'}{\bar{\theta}_1(h_1(u_n(r), M_1 u_n(r)))} \leq \bar{P}'_1(r). \quad (3.10)$$

Integrate (3.10), from 0 to r , we have

$$\int_{a_1}^{u_n(r)} \frac{1}{\bar{\theta}_1(h_1(t, M_1 t))} dt \leq \bar{P}_1(r),$$

which is the same as

$$\mathcal{H}_1(u_n(r)) \leq \bar{P}_1(r). \quad (3.11)$$

Now, we can easily see that \mathcal{H}_1 is a bijection with the inverse function \mathcal{H}_1^{-1} strictly increasing on $[0, \mathcal{H}_1(\infty))$. By combining this with the previous inequality, leads to

$$u_n(r) \leq \mathcal{H}_1^{-1}(\overline{P}_1(r)). \quad (3.12)$$

Returning to $\{v_n(r)\}_{n \geq 0}$, one can show that

$$(v_n(r))' \leq \overline{\theta}_2(h_2(v_n(r), M_2 v_n(r))) \overline{P}_2'(r).$$

Integrating this ordinary differential inequality we get

$$\mathcal{H}_2(v_n(r)) = \int_{a_2}^{v_n(r)} \frac{1}{\overline{\theta}_2(h_2(t, M_2 t))} dt \leq \overline{P}_2(r).$$

From this inequality, we derive

$$v_n(r) \leq \mathcal{H}_2^{-1}(\overline{P}_2(r)). \quad (3.13)$$

In summary, we have found upper bounds for $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ which are dependent of r . Now let us complete the proof of Theorem 3.1. We prove that the sequences $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ are bounded and equicontinuous on $[0, c_0]$ for arbitrary $c_0 > 0$. Indeed, since

$$(u_n(r))' \geq 0 \text{ and } (v_n(r))' \geq 0 \text{ for all } r \geq 0,$$

it follows that

$$u_n(r) \leq u_n(c_0) \leq C_1 \text{ and } v_n(r) \leq v_n(c_0) \leq C_2 \text{ on } [0, c_0].$$

Here $C_1 = \mathcal{H}_1^{-1}(\overline{P}_1(c_0))$ and $C_2 = \mathcal{H}_2^{-1}(\overline{P}_2(c_0))$ are positive constants. Recall that $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ are bounded on $[0, c_0]$ for arbitrary $c_0 > 0$. Using this fact, we show that the same is true of $(u_n(r))'$ and $(v_n(r))'$. Indeed, for any $r \geq 0$,

$$\begin{aligned} (u_n(r))' &= \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1(u_{n-1}(s), v_{n-1}(s)) ds \right) \\ &\leq \Psi_1^{-1} \left(\frac{1}{\xi_1(r)} \int_0^r \xi_1(s) p_1(s) f_1(u_n(s), v_n(s)) ds \right) \\ &\leq \Psi_1^{-1} \left(\|p_1\|_\infty f_1(C_1, C_2) \frac{1}{\xi_1(r)} \int_0^r \xi_1(s) ds \right) \\ &\leq \Psi_1^{-1} \left(\|p_1\|_\infty f_1(C_1, C_2) \int_0^r ds \right) \\ &\leq \Psi_1^{-1} (\|p_1\|_\infty f_1(C_1, C_2) c_0) \text{ on } [0, c_0]. \end{aligned}$$

Similar arguments show that

$$\begin{aligned} (v_n(r))' &= \Psi_2^{-1} \left(\frac{1}{\xi_2(r)} \int_0^r \xi_2(s) p_2(s) f_2(u_{n-1}(s), v_{n-1}(s)) ds \right) \\ &\leq \Psi_2^{-1} (\|p_2\|_\infty f_2(C_1, C_2) c_0) \text{ on } [0, c_0]. \end{aligned}$$

It remains, to prove that $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ are equicontinuous on $[0, c_0]$ for arbitrary $c_0 > 0$. Let $\varepsilon_1, \varepsilon_2 > 0$. To verify equicontinuous on $[0, c_0]$, observe that

$$\begin{aligned} |u_n(x) - u_n(y)| &= |(u_n(\xi_1))'| |x - y| \leq \Psi_1^{-1}(\|p_1\|_\infty f_1(C_1, C_2) c_0) |x - y|, \\ |v_n(x) - v_n(y)| &= |(v_n(\xi_2))'| |x - y| \leq \Psi_2^{-1}(\|p_2\|_\infty f_2(C_1, C_2) c_0) |x - y|, \end{aligned}$$

for all $n \in \mathbb{N}$ and all $x, y \in [0, c_0]$ and for ξ_1, ξ_2 the constants from the mean value theorem. So it suffices to take

$$\delta_1 = \frac{\varepsilon_1}{\Psi_1^{-1}(\|p_1\|_\infty f_1(C_1, C_2) c_0)} \text{ and } \delta_2 = \frac{\varepsilon_2}{\Psi_2^{-1}(\|p_2\|_\infty f_2(C_1, C_2) c_0)},$$

to see that $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ are equicontinuous on $[0, c_0]$. In particular, it follows from the Arzela–Ascoli theorem that there exists a function $u \in C([0, c_0])$ and a subsequence N_1 of \mathbb{N}^* with $u_n(r)$ converging uniformly on u to $[0, c_0]$ as $n \rightarrow \infty$ through N_1 . By the same token there exists a function $v \in C([0, c_0])$ and a subsequence N_2 of \mathbb{N}^* with $v_n(r)$ converging uniformly to v on $[0, c_0]$ as $n \rightarrow \infty$ through N_2 . Thus $\{(u_n(r), v_n(r))\}_{n \in N_2}$ converges uniformly on $[0, c_0]$ to $(u, v) \in C([0, c_0]) \times C([0, c_0])$ through N_2 (see Lü-O'Regan-Agarwal [17]). The limit function (u, v) constructed in this way will be nonnegative, radially symmetric and nondecreasing with respect to r and is a solution of system (1.1). Moreover, the radial solutions of (1.1) with $u(0) = a_1, v(0) = a_2$ satisfy:

$$u(r) = a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(u(s), v(s)) ds \right) dt, \quad r \geq 0, \quad (3.14)$$

$$v(r) = a_2 + \int_0^r \Psi_2^{-1} \left(\frac{1}{\xi_2(t)} \int_0^t \xi_2(s) p_2(s) f_2(u(s), v(s)) ds \right) dt, \quad r \geq 0. \quad (3.15)$$

In the case $\underline{P}_1(\infty) = \underline{P}_2(\infty) = \infty$, we observe that

$$\begin{aligned} u(r) &= a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(u(s), v(s)) ds \right) dt \\ &\geq a_1 + \int_0^r \Psi_1^{-1} \left(\frac{1}{\xi_1(t)} \int_0^t \xi_1(s) p_1(s) f_1(a_1, a_2 + \underline{\theta}_2(f_2(a_1, a_2)) P_2(s)) ds \right) dt \\ &= a_1 + \underline{P}_1(r). \end{aligned}$$

We repeat the argument applied in the proof of (3.16)

$$\begin{aligned} v(r) &= a_2 + \int_0^r \Psi_2^{-1} \left(\frac{1}{\xi_2(t)} \int_0^t \xi_2(s) p_2(s) f_2(u(s), v(s)) ds \right) dt \\ &\geq a_2 + \int_0^r \Psi_2^{-1} \left(\frac{1}{\xi_2(t)} \int_0^t \xi_2(s) p_2(s) f_2(a_1 + \underline{\theta}_1(f_2(a_1, a_2)) P_1(s), a_2) ds \right) dt \\ &= a_2 + \underline{P}_2(r). \end{aligned}$$

By taking limits in (3.16) and (3.17), we get entire large solutions

$$\lim_{r \rightarrow \infty} u(r) = \infty \text{ and } \lim_{r \rightarrow \infty} v(r) = \infty.$$

Consequently, (u, v) is an entire large solution of (1.1).

The next purpose of the paper is to give a sufficient condition to obtain an entire bounded solution to (1.1). Our result in this case is the following:

Theorem 3.2. *The system (1.1) has one positive radial solution $(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$ given that $\mathcal{H}_1(\infty) = \mathcal{H}_2(\infty) = \infty$ and (P1), (C1), (C2) hold true. Moreover, if $\overline{P}_1(\infty) < \infty$ and $\overline{P}_2(\infty) < \infty$ then*

$$\lim_{r \rightarrow \infty} u(r) < \infty \text{ and } \lim_{r \rightarrow \infty} v(r) < \infty.$$

Proof of Theorem 3.2: The existence part is proved in Theorem 3.1. Assume $\overline{P}_1(\infty) < \infty$ and $\overline{P}_2(\infty) < \infty$. Proceeding as in the proof of (3.12) and (3.13) with the integral equations (3.14) and (3.15), one gets the estimates

$$u(r) \leq \mathcal{H}_1^{-1}(\overline{P}_1(\infty)) < \infty \text{ and } v(r) \leq \mathcal{H}_2^{-1}(\overline{P}_2(\infty)) < \infty \text{ for all } r \geq 0.$$

Thus (u, v) is a positive entire bounded solution of the system (1.1).

Concerning the existence of semifinite entire large solutions to (1.1), we have the following:

Theorem 3.3. *The system (1.1) has one positive radial solution $(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$ given that $\mathcal{H}_1(\infty) = \mathcal{H}_2(\infty) = \infty$ and (P1), (C1), (C2) hold true. Moreover, the following hold:*

1) *If $\overline{P}_1(\infty) < \infty$ and $\underline{P}_2(\infty) = \infty$ then*

$$\lim_{r \rightarrow \infty} u(r) < \infty \text{ and } \lim_{r \rightarrow \infty} v(r) = \infty.$$

2) *If $\underline{P}_1(\infty) = \infty$ and $\overline{P}_2(\infty) < \infty$ then*

$$\lim_{r \rightarrow \infty} u(r) = \infty \text{ and } \lim_{r \rightarrow \infty} v(r) < \infty.$$

Proof of Theorem 3.3: The existence part is proved in Theorem 3.1.

1): As in the proof of Theorem 3.1 and Theorem 3.2, we have

$$u(r) \leq \mathcal{H}_1^{-1}(\overline{P}_1(\infty)) < \infty \text{ and } v(r) \geq a_2 + \underline{P}_1(r).$$

Observing that $\overline{P}_1(\infty) < \infty$ and $\underline{P}_2(\infty) = \infty$ the above relations yield

$$\lim_{r \rightarrow \infty} u(r) < \infty \text{ and } \lim_{r \rightarrow \infty} v(r) = \infty.$$

This completes the proof.

2): Arguing as above, we obtain

$$u(r) \geq a_1 + \underline{P}_1(r) \text{ and } v(r) \leq \mathcal{H}_2^{-1}(\overline{P}_2(r)). \quad (3.18)$$

Our conclusion follows now by letting $r \rightarrow \infty$ in (3.18).

We now propose a more refined question concerning the solutions of system (1.1). In analogy with Theorems 3.1-3.3, we can also prove the following three theorems. The first is the following:

Theorem 3.4. *The system (1.1) has one positive bounded radial solution $(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$ given that $\overline{P}_1(\infty) < \mathcal{H}_1(\infty) < \infty$, $\overline{P}_2(\infty) < \mathcal{H}_2(\infty) < \infty$, (P1), (C1), (C2) hold true. Moreover,*

$$\begin{cases} a_1 + \underline{P}_1(r) \leq u(r) \leq \mathcal{H}_1^{-1}(\overline{P}_1(r)), \\ a_2 + \underline{P}_1(r) \leq v(r) \leq \mathcal{H}_2^{-1}(\overline{P}_2(r)). \end{cases}$$

Proof of Theorem 3.4: The existence part is proved in Theorem 3.1. Next, by a simple calculation together with (3.11) and the conditions of the theorem we obtain:

$$\mathcal{H}_1(u_n(r)) \leq \overline{P}_1(\infty) < \mathcal{H}_1(\infty) < \infty \text{ and } v_n(r) \leq \mathcal{H}_2^{-1}(\overline{P}_2(\infty)) < \infty.$$

On the other hand, since \mathcal{H}_1^{-1} is strictly increasing on $[0, \mathcal{H}_1(\infty))$, we find that

$$u_n(r) \leq \mathcal{H}_1^{-1}(\overline{P}_1(\infty)) < \infty,$$

and then the non-decreasing sequences $\{u_n\}_{n \geq 0}$ and $\{v_n\}_{n \geq 0}$ are bounded above for all $r \geq 0$ and all n . Now we use this observation to conclude

$$(u_n(r), v_n(r)) \xrightarrow{n \rightarrow \infty} (u(r), v(r))$$

and then the limit functions u and v are positive entire bounded radial solutions of system (1.1). This completes the proof.

Theorem 3.5. *Assume (P1), (C1) and (C2) hold true. The following hold true:*

i) *The system (1.1) has one positive radial solution $(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$ such that $\lim_{r \rightarrow \infty} u(r) = \infty$ and $\lim_{r \rightarrow \infty} v(r) < \infty$ given that $\mathcal{H}_1(\infty) = \infty$, $\underline{P}_1(\infty) = \infty$ and $\overline{P}_2(\infty) < \mathcal{H}_2(\infty) < \infty$.*

ii) *The system (1.1) has one positive radial solution $(u, v) \in C^1([0, \infty)) \times C^1([0, \infty))$ such that $\lim_{r \rightarrow \infty} u(r) < \infty$ and $\lim_{r \rightarrow \infty} v(r) = \infty$ given that $\overline{P}_1(\infty) < \mathcal{H}_1(\infty) < \infty$ and $\mathcal{H}_2(\infty) = \infty$, $\underline{P}_2(\infty) = \infty$.*

Proof of Theorem 3.5: The proof for these cases is similar as the above and is therefore omitted.

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